

Conformal invariance in driven diffusive systems at high currents

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We consider space-time correlations in driven diffusive systems which undergo a fluctuation into a regime with an atypically large current or dynamical activity. For a single conserved mass we show that the spatio-temporal density correlations in one space dimension are given by conformally invariant field theories with central charge $c = 1$, corresponding to a ballistic universality class with dynamical exponent $z = 1$. We derive a phase diagram for atypical behaviour that besides the conformally invariant regime exhibits a regime of phase separation for atypically low current or activity. On the phase transition line, corresponding to typical behaviour, the dynamics belongs to the Kardar-Parisi-Zhang universality class with dynamical exponent $z = 3/2$, except for a diffusive point with $z = 2$. We demonstrate the validity of the theory for the one-dimensional asymmetric simple exclusion process with both periodic and open boundaries by exact results for the dynamical structure in the limit of maximal current.

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An intriguing question is whether a many-body system far from thermal equilibrium that undergoes some big atypical fluctuation can be understood in terms of an upscaled description of “normal” spatio-temporal behaviour or whether it behaves in a qualitatively different fashion during this atypical fluctuation. The statistical properties of such fluctuations are also the fundamental object of interest in large deviation theory which provides – in equilibrium – the foundations of thermodynamics in terms of thermodynamic ensembles. Of course, one does not expect a unique answer to a question posed so generally, but some insight into large fluctuations without external trigger may be gained from considering a reasonably wide, but still well-defined class of model systems, viz. stochastic lattice gas models that have served as paradigmatic models for non-equilibrium phenomena in the last decades [1–3].

These models have a stochastic particle hopping dynamics that mimics noise, interactions between particles frequently include a hard-core repulsion, and a bulk driving field or boundary gradients maintain a fluctuating non-equilibrium steady state with a non-zero locally conserved mass-current. The paradigmatic example is the asymmetric simple exclusion process (ASEP) in one dimension [1–4] where a lattice site can be occupied by at most one particle and particles jump randomly to their nearest neighbour sites, provided the target site is empty. A bias with jump rates p to the right and q to the left and/or open boundaries where particle exchanges with reservoirs take place create a non-equilibrium situation.

The question is whether in a driven lattice gas the spatio-temporal fluctuation patterns remain essentially unchanged during an untypical large fluctuation of the current (or more generally of the undirected jump activity of the particles), or whether correlations during such

a fluctuation are qualitatively different from typical behaviour, which in one space dimension has recently been shown to be universal and within the scope of the theory of non-linear fluctuating hydrodynamics [5] which predicts for one conserved current either diffusive behaviour with dynamical critical exponent $z = 2$ or fluctuations in the universality class of the Kardar-Parisi-Zhang (KPZ) equation with $z = 3/2$.

The answer that we shall give is, in a nutshell, the following: If a system with hard-core repulsion that is typically in a stationary state of spatially homogeneous density undergoes a large fluctuation into a regime of *low* current or jump activity then this is most likely realized not by fluctuations in the random time after which particles attempt to jump (upscaling), but by spontaneous phase separation into spatial domains of high and low density respectively. Conversely, during a fluctuation into a regime of *high* current or activity the homogeneous stationary density is maintained, but a qualitative change of correlations between particle positions makes the atypical large fluctuation least unlikely and hence typically realizes it. More specifically, we assert that these space-time correlations are universal and can be predicted for non-equilibrium particle systems in one space dimension from conformal field theory for two-dimensional equilibrium critical phenomena [6, 7].

The first answer concerning atypically low current or activity can be understood by noting that in a region of high density fewer jumps will occur naturally due to the repulsive interaction, while in a low-density region fewer jumps occur trivially because of the smaller number of particles, thus optimizing the probability for such untypical behaviour. This picture is well borne out by the powerful machinery of Macroscopic Fluctuation Theory [8] which demonstrates for the ASEP with periodic

boundary conditions that phase separation sets in below some critical atypical current [9, 10]. For open boundary conditions a similar phenomenon occurs [11–13]. Such a dynamical phase transition was found also for an atypically low activity in the symmetric simple exclusion process (SSEP) which has no hopping bias [14].

On the contrary, for a fluctuation into a regime of high current or activity phase separation would be counter-productive and one expects a homogeneous bulk density as in the typical steady state. However, no theoretical framework for a quantitative description of this regime, which is inaccessible to macroscopic fluctuation theory, has been given yet. It is aim of this work to establish that in 1+1 dimensions this regime can be described by conformal field theory (CFT).

More precisely, we predict that in the stationary regime of an atypical speeding-up of particle hopping in an interacting system with a single locally conserved current the dynamical structure function $S(k, l, t) = \langle (n_k(t) - \rho_k)(n_l(t) - \rho_l) \rangle$, i.e., the stationary space-time correlations of the local particle numbers $n_k(t)$ with local average density ρ_k , has in a translation invariant setting with $x = k - l$ the universal scaling form

$$S(x, t) = \frac{C_1}{v_L^2 t^2} \frac{1 - \xi^2}{(1 + \xi^2)^2} + \frac{C_2}{(v_L t)^{2\gamma}} \frac{\cos[2(q^* x - \omega t)]}{(1 + \xi^2)^{2\gamma}} \quad (1)$$

with the scaling variable $\xi = (x - v_c t)/(v_L t)$ indicating dynamical exponent $z = 1$ rather than 2 or 3/2 for typical dynamics. The static critical exponent $\gamma \geq 1$ depends on the particle interactions, the collective velocity v_c , the Luttinger liquid time scale v_L , the wave vector q^* , and the oscillation frequency ω can be computed from the generally complex dispersion relation as discussed below, and C_i are non-universal amplitudes that depend on the microscopic details of the model. Higher-order correlations are fully determined by a modified CFT with central charge $c = 1$ of the Virasoro algebra, the difference to usual conformal invariance being the appearance of the collective velocity v_c in the Galilei-shift of the space coordinate x and the time-dependence of the oscillating part of the correlation function with frequency ω .

We prove this assertion for the ASEP (Fig. (1)) in the regime of maximal current, both for periodic and open boundary conditions, by using the exact mapping of the dynamics of the ASEP conditioned on an atypical current or hopping activity to a non-Hermitian Heisenberg spin-1/2 quantum chain in the ferromagnetic range with anisotropy parameter $\Delta \geq 0$ and imaginary Dzyaloshinskii–Moriya interaction [15]

$$\tilde{H} = -\frac{w}{2} \sum_{k=1}^L \tilde{g}_k \quad (2)$$

with

$$\tilde{g}_k = 2v\sigma_k^+ \sigma_{k+1}^- + 2v^{-1}\sigma_k^- \sigma_{k+1}^+ + \Delta(\sigma_k^z \sigma_{k+1}^z - 1). \quad (3)$$

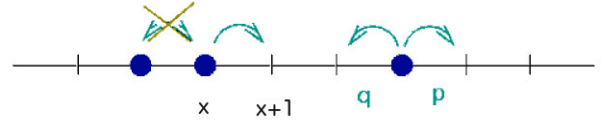


Figure 1: (Color online) Schematic representation of the asymmetric simple exclusion process. A particle hops to the neighbouring site provided this target site is empty. In the regime of maximal current the jumps to the left do not contribute to the statistical properties of the conditioned process.

Here

$$e^f = \sqrt{\frac{p}{q}}, \quad w = \sqrt{pq}e^\mu, \quad v = e^{f+\lambda}, \quad \Delta = e^{-\mu} \cosh f \quad (4)$$

and λ is conjugate to the current j and μ is conjugate to the activity a on which we condition. The case $\lambda = \mu = 0$ corresponds to the typical dynamics with stationary current $j = (p - q)\rho(1 - \rho)$ and activity $a = (p + q)\rho(1 - \rho)$ for particle density $\rho = N/L$. Positive λ and μ correspond to high current and activity resp. [16, 17].

The ground state of the *Hermitian* spin-1/2 Heisenberg quantum chain is known to be described by CFT with central charge $c = 1$ [6, 7, 18]. In order to show that the non-Hermitian terms in (3) lead to the modifications of CFT described above we focus on maximal current $\lambda \rightarrow \infty$. Following [19, 20], the maximal current is realized by two mechanisms: The trivial speed-up of the jump frequency and a non-trivial building up of correlations. In order to extract the non-trivial part we rescale time by $pe^{\lambda+\mu}$ so that we are left with

$$H = -\sum_{k=1}^L \sigma_k^+ \sigma_{k+1}^- \quad (5)$$

for periodic boundary conditions with L sites.

The dynamical structure function in this limit was computed in [20] and represents a ballistic dynamical universality class with dynamical exponent $z = 1$ that was first studied by Spohn [21]. The large-scale behaviour is given by the non-oscillating part in (1). However, neither in [20] nor in [21] the oscillating contribution has been pointed out. In order to extract this oscillating part and uncover other hallmarks of conformal invariance we take the standard approach by diagonalizing (5) in terms of Jordan-Wigner fermionic operators [22]. Fourier transformation with momentum p then yields

$$H = \sum_{p=1}^L (\mathcal{P}^- \epsilon_p \hat{c}_p^\dagger \hat{c}_p + \mathcal{P}^+ \epsilon_{p-\frac{1}{2}} \hat{c}_{p-\frac{1}{2}}^\dagger \hat{c}_{p-\frac{1}{2}}) \quad (6)$$

in the graded Fock space build by the free fermion creation operators \hat{c}_p^\dagger and annihilation operators \hat{c}_p in the

even particle sector and $\hat{c}_{p-1/2}^\dagger, \hat{c}_{p-1/2}$ in the odd sector. The single-particle energy is

$$\epsilon_p := -e^{-\frac{2\pi ip}{L}}. \quad (7)$$

Remarkably the ground state for $0 \leq N \leq [L/2]$ particles, which is defined by having the lowest real part of the eigenvalues of (6), is the same as for the Hermitian case with ground state energy per site given by

$$-e_0 = \frac{v_F}{L \sin(\pi/L)} = v_F \left(\frac{1}{\pi} + \frac{\pi}{6L^2} + O(L^{-4}) \right) \quad (8)$$

where $v_F = \sin(\pi\rho)$ is the Fermi velocity. The energy gaps, however, are in general complex. The lowest gap (with smallest real part) is given by $\delta = \epsilon_{(N+1)/2} - \epsilon_{(N-1)/2} = 2 \sin(\pi/L)(\sin(\pi\rho) + i \cos(\pi\rho))$ with real part

$$\Re(\delta) = v_F \frac{2\pi}{L} (1 + O(L^{-2})). \quad (9)$$

For the dynamical structure function at density ρ the Wick theorem gives

$$S_\rho(n, m, t) = \langle c_n^\dagger(t) c_m(0) \rangle \langle c_n(t) c_m^\dagger(0) \rangle - \langle c_n^\dagger(t) c_m^\dagger(0) \rangle \langle c_n(t) c_m(0) \rangle \quad (10)$$

where the second part vanishes in the periodic system due to particle number conservation. The time-dependent operators are defined by $X(t) := e^{Ht} X e^{-Ht}$ which leads to $\hat{c}_p(t) = e^{-\epsilon_p t} \hat{c}_p$ and $\hat{c}_p^\dagger(t) = e^{\epsilon_p t} \hat{c}_p^\dagger$ and allows us to compute straightforwardly the basic correlators in (10).

In terms of the function

$$f_{L,N}(r, t) := \begin{cases} \frac{1}{L} \sum_{p=-(N-1)/2}^{(N-1)/2} e^{\frac{-2\pi i}{L} pr + \epsilon_p t} & N \text{ odd} \\ \frac{1}{L} \sum_{p=-N/2+1}^{N/2} e^{\frac{-2\pi i}{L} (p-\frac{1}{2})r + \epsilon_{p-\frac{1}{2}} t} & N \text{ even.} \end{cases} \quad (11)$$

we find the exact result

$$\langle c_{n+r}^\dagger(t) c_n(0) \rangle_{L,N} = f_{L,N}(r, t) \quad (12)$$

$$\langle c_{n+r}(t) c_n^\dagger(0) \rangle_{L,N} = (-1)^r f_{L,N}(-r, t) \quad (13)$$

and therefore

$$S_\rho(r, t) = (-1)^r f_{L,N}(r, t) f_{L,N}(-r, t). \quad (14)$$

Now we study the thermodynamic limit $L, N \rightarrow \infty$ such that $\rho = N/L$ is finite. We recall the Fermi momentum $k_F = \pi\rho$ and introduce the continuum dispersion relation $\epsilon(p) := -e^{-ip}$ with its real and imaginary parts $\epsilon_1(p) := \Re(\epsilon(p)) = -\cos p$, $\epsilon_2(p) := \Im(\epsilon(p)) = \sin p$. By taking the derivative of the continuum dispersion relation w.r.t. the momentum p one obtains the Fermi velocity

$v_F = \epsilon'_1(k_F) = \sin k_F = \sin(\pi\rho)$ and the collective velocity $v_c = \epsilon'_2(k_F) = \cos k_F = \cos(\pi\rho)$. Notice that in the Hermitian case $\epsilon_2(p) = 0$ and therefore $v_c = 0$.

We also define the complex coordinate

$$z := ir + \epsilon'(k_F)t = i\tilde{r} + v_F t = v_F t(1 + i\xi) \quad (15)$$

where $\tilde{r} = r - v_c t$ and the phase angle

$$\varphi(r, t) := k_F r - \epsilon_2(k_F)t = k_F r - \sin(k_F)t. \quad (16)$$

Scaling analysis of (11) then yields

$$f_\rho(r, t) = \frac{e^{-i\varphi(r, t) - t \cos(k_F)}}{2\pi \bar{z}} + c.c. \quad (17)$$

and we arrive at

$$S_\rho(r, t) = \frac{1}{4\pi^2} \left(\frac{1}{z^2} + \frac{1}{\bar{z}^2} + \frac{2 \cos(2\varphi(r, t))}{z\bar{z}} \right) \quad (18)$$

$$= \frac{1}{2(\pi v_F t)^2} \left[\frac{1 - \xi^2}{(1 + \xi^2)^2} + \frac{\cos(2\varphi(r, t))}{1 + \xi^2} \right] \quad (19)$$

which is of the predicted form (1) with $v_L = v_F$ and shows that in the Hermitian case where $\epsilon_2(p) = 0$ one has $v_c = \omega = 0$ as in usual CFT.

Next we consider open boundary conditions. For maximal positive current the back-hopping rates proportional to q are irrelevant and only injection to site 1 with rate αp and absorption into a reservoir at site L with rate βp need to be considered. Nevertheless, open boundaries create several technical difficulties for the exact treatment, viz. lack of periodicity, violation of particle conservation and loss of the bilinear free-fermion property underlying the computations of the previous paragraphs. The latter problem, however, can be overcome by augmenting the lattice with auxiliary boundary sites 0 and $L+1$ which swap their state (empty or occupied) whenever a creation or annihilation event occurs. This leaves the dynamics of the exclusion process unchanged.

The dependence on the boundary rates α and β can be removed by the similarity transformation $V = \prod_{k=1}^L u_k^{\hat{n}_k}$ with the choice $p = (2\alpha\beta)^{-\frac{1}{L+1}}$ and $u_k = \sqrt{2\alpha} p^k$. The transformed generator then reads in terms of Jordan-Wigner fermions

$$H = - \sum_{k=1}^{L-1} c_{k+1}^\dagger c_k - \frac{1}{\sqrt{2}} \left[c_1^\dagger (c_0 - c_0^\dagger) + (c_{L+1} + c_{L+1}^\dagger) c_L \right]. \quad (20)$$

For L even the lack of periodicity and particle number conservation is overcome by the Bogolyubov transformation

$$b_k = \frac{1}{\sqrt{2}} \left(c_k + (-1)^k c_{L+1-k}^\dagger \right), \quad 1 \leq k \leq L \quad (21)$$

$$b_0 = \frac{1}{2} \left(c_0 - c_0^\dagger + c_{L+1} + c_{L+1}^\dagger \right). \quad (22)$$

The equations of motion read $[H, b_k] = b_{k-1}$ with periodic boundary conditions, even though the original problem has no translation invariance. Subsequent Fourier transformation leads to $H = \sum_{p=0}^L \epsilon_p \hat{b}_p^\dagger \hat{b}_p$ with the single-particle “energies” $\epsilon_p = -e^{-\frac{2\pi i p}{L+1}}$.

Thus we are back to the periodic problem, albeit with $L+1$ sites and only odd particle number which follows from the boundary conditions in (20). The ground state is given by populating all negative energy modes $\Re(\epsilon_p) = \cos(2\pi p/(L+1)) \leq 0$ and therefore

$$-e_0^{open} = \frac{1}{\pi} + \frac{1}{\pi L} + \frac{\pi}{24L^2} + O(L^{-3}). \quad (23)$$

For the lowest energy gap one finds the leading finite-size correction for the real part

$$\Re(\delta^{open}) = \frac{4\pi}{L} + O(L^{-2}) \quad (24)$$

which is twice the value of the periodic system at half-filling where $v_F = 1$.

The stationary density of the conditioned process is constant with $\rho_n = 1/2$ as one would expect for maximal current or activity. For the dynamical structure function one can repeat the analysis of the periodic system due to the pseudo-periodicity of (21). However, we need the full Wick theorem since the second part is non-zero because of the lack of particle number conservation. After some computation using the Bogolyubov transformation (21) and taking the thermodynamic limit one arrives at

$$S^{open}(n, m, t) = S_{1/2}(n - m, t) - S_{1/2}(n + m, t). \quad (25)$$

Intriguingly the ground state results (8) and (9) for the periodic system and (23) and (24) for the open system can be understood in terms of CFT even though (6) and (20) are strongly non-Hermitian. After rescaling time by the Fermi velocity v_F (identified as such by the computation of the dynamical structure function) and using finite-size scaling theory for conformal invariance [23, 24] the leading corrections (8) and (23) to the ground state energy correspond to central charge $c = 1$ of the Virasoro algebra. The real part $2\pi x/L$ of the lowest energy gap corresponds to the lowest critical bulk exponent $x = 1$ of a primary field of the corresponding CFT for the periodic case and $x = 2$ for the open boundary conditions, in complete analogy to the CFT describing the Hermitian XX -chain at half-filling $\rho = 1/2$. The non-hermitian nature of the time evolution enters only through the imaginary part of the energy gap which yields the collective velocity $v_c = \cos(\pi\rho)$ and the oscillation frequency $\omega = \sin(\pi\rho)$.

The same CFT describes other quantum spin chains such as the spin- s Heisenberg chain in the gapless regime [25, 26] with anisotropy $0 \leq \Delta < 1$ whose non-hermitian generalization maps to the partial exclusion process [27] conditioned an atypical current and activity. One has an energy gap for integer spin (Haldane conjecture) only

for negative Δ . In fact, it opens up exactly at $\Delta = 0$ where the mapping to stochastic dynamics gets lost, as positivity of the transition rates implies $\Delta \geq 0$. The energy gap is finite in the ferromagnetic regime $\Delta > 1$, corresponding to phase separation in the particle system. These observations support the notion of universality of (1) and also confirm the dynamical phase transition to a phase-separated regime at low current or activity.

The dynamical structure function can be understood in terms of CFT by adapting the standard splitting of the free-fermion operators $c_n(t)$ into right-movers (with positive momentum) and left-movers (with negative momentum) in the hermitian case [28] to the non-hermitian scenario. With $\varphi(n, t) = k_F n - \Im(\epsilon_{k_F})t$ and $\varepsilon = \Re(\epsilon_{k_F})$ this yields

$$c_n(t) = \frac{1}{\sqrt{2\pi}} \left[e^{-i\varphi(n,t)+\varepsilon t} \psi_{\tilde{n}}(t) + e^{i\varphi(n,t)-\varepsilon t} \bar{\psi}_{\tilde{n}}(t) \right] \quad (26)$$

and similar for $c_n^\dagger(t)$ with Galilei-transformed coordinate $\tilde{n} = n - v_c t$. For large n and t the two-point functions are predicted from CFT with central charge $c = 1$ to be

$$\langle \psi^\dagger(z) \psi(z') \rangle = \frac{1}{z - z'}, \quad \langle \bar{\psi}^\dagger(\bar{z}) \bar{\psi}(\bar{z}') \rangle = \frac{1}{\bar{z} - \bar{z}'}. \quad (27)$$

This then leads to the dynamical structure function with static exponent $\gamma = 1$ that we computed. The same scaling form (1) with a non-universal exponent $\gamma > 1$ can be obtained via bosonization [28] for non-maximal speeding-up, i.e., $\Delta > 0$. Since t is to be understood in units of the lattice spacing the oscillating part thus becomes irrelevant even without spatial coarse-graining. The dynamical structure function for open boundaries can be understood from boundary conformal field theory with reflecting boundary conditions which yields non-vanishing correlators between the right- and left-movers which are of the form similar to (27), but with a $1/(z+z')$ dependence coming from reflection.

From (3) we read off the remarkable fact that for $\Delta = \cosh(f + \lambda)$, i.e., at a critical activity parameter

$$\mu_c = -\ln \left(\frac{\cosh(f + \lambda)}{\cosh f} \right) \quad (28)$$

the conditioned generator becomes the generator of an unconditioned ASEP with hopping asymmetry $f' = f + \lambda$. Then for $\mu < \mu_c$ one has phase separated atypical behaviour as shown in [14] for $f = \lambda = 0$ and in [21] for $f = 0$, whereas for $\mu > \mu_c$ one has – as elaborated above – conformal invariance. On the phase transition line $\mu = \mu_c$ one has typical stochastic dynamics with $z = 2$ for $\lambda = -f$ (symmetric simple exclusion process) and $z = 3/2$ for $\lambda \neq -f$ (ASEP).

Generally we argue that the phase diagram of atypical behaviour in d -dimensional driven diffusive systems far from thermal equilibrium can be explored by studying the critical behaviour of $d+1$ -dimensional equilibrium models. In $1+1$ dimension with a single conserved

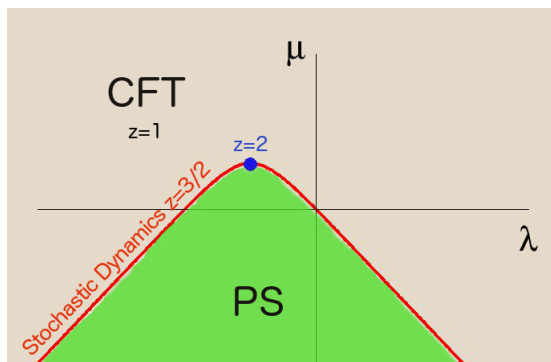


Figure 2: (Color online) Phase diagram of the conditioned ASEP as a function of current parameter λ and the activity parameter μ for positive drive $f > 0$. The exact phase transition line that separates the conformally invariant regime CFT from the phase-separated regime PS is given by (28).

species there is a dynamical phase transition from a dynamical critical regime for fast dynamics (described by conformal field theory) through a phase transition line corresponding to typical dynamics (described by nonlinear fluctuating hydrodynamics) to a phase-separated regime of slow dynamics (accessible by macroscopic fluctuation theory). This demonstrates that large fluctuations in driven diffusive systems cannot be understood by some simple upscaling procedure but is sustained by spatio-temporal correlations that are qualitatively different from typical behaviour.

For systems with more than one conservation law the scenario is more complex. It has been shown recently that typical behaviour of stochastic particle systems with local dynamics have dynamical critical exponents $z_i > 1$ given by the ratios of neighboring Fibonacci numbers F_{i+1}/F_i , starting with $F_2 = 1$ and $F_3 = 2$ [29]. Also, quantum systems for n conserved species that would describe the critical behaviour in the regime of high current or activity may be governed by a CFT with central charge $c > 1$. For low current or activity one still expects a phase-separated regime, which, however, may exhibit richer behaviour than for $n = 1$.

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